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Theoretical and experimetal analyses of the maximum-suppotable fluid load on a rotating cylinder

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Abstract. Thin-film theory is used to derive an implicit criterion for the existence of a viscous, free-surface flow on the outer surface of a rotating circular cylinder. We estimate both the maximum fluid load supportable at a given Stokes number, and the maximum Stokes number permitting the support of a prescribed fluid load. The results of our theory are in excellent agreement with those obtained from an independent, numerical solution of the full Stokes equations; the comparison reveals that our thin-film theory works well beyond its expected range of validity.
Explicit existence criteria, in the form of small-parameter expansions, are presented for the extreme cases of small fluxes and large Stokes numbers. We derive an expression for the minimum angular velocity which will support a given load; our results show qualitative agreement with those of basic experiments.

1. Introduction

If a cylinder is covered in a thin film of viscous liquid and subsequently rotated about a horizontal axis, intuition suggests that a steady-state, two-dimensional, free-surface profile can be obtained provided the rotation rate is sufficiently large. Conversely, if the rotation rate is too low, it is obvious that some of the liquid will drain off, whereafter a steady-state profile with the reduced mass of fluid might be achieved.

Pukhnachev [1, 2] addressed the general question of the existence of steady solutions to the Navier-Stokes equations for flows bounded in part by a solid boundary and in part by a free surface. Inspired by Pukhnachev, Moffatt [3] used lubrication theory to examine the two-dimensional flow of a viscous thin film on a rotating cylinder. Preziosi and Joseph [4] addressed the axial variation in the flow by means of experiments. Hansen and Kelmanson [5] studied the two-dimensional problem using an integral-equation method which can be applied to flows in which the thickness of the fluid layer can be of the same order as the cylinder radius; such flows are not amenable to the analyses of either Moffatt [3] or Preziosi and Joseph [4], who discard a first-order term in their theory on the grounds that it is negligible for thin-film flows. On this basis, Moffatt [3] was able to derive an approximate existence criterion for a solution to the problem so described.

In the present paper, we include the aforementioned first-order term in an effort to establish not only a more accurate existence criterion, but also an improved maximum-load analysis. Following the necessary introductory thin-film theory of section 2, section 3 reveals that the inclusion of the first-order term complicates substantially the derivation of the refined existence criterion. Based upon a comparison with the integral-equation results (for the *full* Stokes approximation) of Hansen and Kelmanson [5], it is shown that, for a given Stokes number, the present theory slightly overestimates the maximum fluid flux which can exist in the flow on the cylinder. It is observed that this overestimation is almost constant, and so we are able to make a simple adjustment to our existence criterion. Section 4 presents a maximum-load analysis, the results of which are in excellent agreement with the integral-equation results of Hansen and Kelmanson [5]. In section 5, we examine the converse problem of the minimum cylinder rotation rate which can occur before the onset of fluid draining. Section 6 describes an experimental investigation of the theory of section 5, and qualitative agreement is observed.

The work described herein has a realistic practical application in the roller-coating industry [6]; it is not just an esoteric exercise. Both the theoretical and numerical results reported below should enable predictions to be made of the maximum possible film thickness attainable before the expected onset of surface instabilities; such a knowledge is of real value in the industrial creation of thin *uniform* films, e.g., as required in the production of photographic film and aluminium foil.

2. Thin-film theory

Flow is assumed to be steady and two dimensional, in an annular region bounded within, by a rotating solid circular cylinder, and without, by a free surface. There is no component of flow parallel with the cylinder axis. The relevant physical parameters are: fluid density, ρ; acceleration due to gravity, g; cylinder radius, a; angular velocity of the cylinder, ω; coefficient of dynamic viscosity, μ. In the subsequent analysis, all variables are to be understood as having been non-dimensionalized with respect to these parameters.

Following Preziosi and Joseph [4], we neglect variations in the radial component of fluid velocity and solve the thin-film approximation to the Navier-Stokes equations in standard polar coordinates (r, ϑ), namely

$$u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u = \gamma \cos\vartheta. \tag{2.1}$$

In (2.1), the non-dimensional parameter γ is the Stokes number, given by

$$\gamma = \frac{\rho g a}{\omega \mu}. \tag{2.2}$$

The approximate condition expressing the vanishing of shear stress on the free surface is

$$\frac{\partial}{\partial r}\left(\frac{u}{r}\right) = 0 \quad \text{on} \quad r = 1 + h(\vartheta), \tag{2.3}$$

and the no-slip condition on the cylinder surface is

$$u = 1 \quad \text{on} \quad r = 1. \tag{2.4}$$

In (2.2), $h(\vartheta)$ is the height of the free surface above the cylinder. We now solve (2.1) subject to (2.3) and (2.4), substitute $1 + \eta$ for r $(0 < \eta \ll 1)$, and expand in powers of η. Neglecting cubic and higher order terms in η and h, we obtain the tangential velocity distribution

$$u(1 + \eta, \vartheta) = 1 + \eta - \frac{\gamma \cos\vartheta}{2}\eta(2h - \eta). \tag{2.5}$$

Eqn (2.5) is given, in different notation, as eqn (3.2) of Preziosi and Joseph [4], who proceed to neglect the η term on the basis that $0 < \eta \ll 1$. Moffat [3] omits the η term by replacing the free-surface boundary condition (2.3) with

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad r = 1 + h(\vartheta). \tag{2.6}$$

In the lubrication approximation, $0 < h \ll 1$ and, in the steady state, the total fluid flux, Q, across the fluid film is a constant given by

$$Q = \int_0^h u\,\mathrm{d}\eta = h + \frac{1}{2}h^2 - \frac{\gamma\cos\vartheta}{3}h^3. \tag{2.7}$$

The tangential velocity at the free surface, $v(\vartheta)$, is given by (2.5) as

$$v(\vartheta) = u(1+h,\vartheta) = 1 + h - \frac{\gamma\cos\vartheta}{2}h^2. \tag{2.8}$$

From (2.7), Q, as a function of h, is a maximum when $\vartheta = 0$ and $h = h_1$, where h_1 satisfies $h_1^2 = (1+h_1)/\gamma$. Since $0 < h_1 \ll 1$, this requires that $\gamma \gg 1$.

Since Q is a constant, eqn (2.7) expresses the fact that $h(\vartheta)$ can be only weakly dependent on ϑ. Expanding h as a power series in Q and equating coefficients therefore yields

$$h(Q,\vartheta) = Q - \frac{1}{2}Q^2 + \left[\frac{\gamma\cos\vartheta}{3} + \frac{1}{2}\right]Q^3. \tag{2.9}$$

Hence, to the extent that lubrication theory is applicable, h is indeed weakly dependent on ϑ, such variation as there is being only third order in Q. If $\hat{Q} = \hat{Q}(\gamma)$ is the maximum flux which can exist for a given Stokes number γ, (2.9) gives the maximum fluid load supportable, $\hat{W}$, in that case by

$$\hat{W} \equiv \int_0^{2\pi} h(\hat{Q},\vartheta)\,\mathrm{d}\vartheta = \hat{Q}(\hat{Q}^2 - \hat{Q} + 2)\pi. \tag{2.10}$$

In the next section, we derive the conditions under which the problem, as described, has a solution.

3. An existence condition for steady flow

Eqns (2.7) and (2.8) may be used to eliminate the cubic term in h:

$$Q = \frac{1}{3}h(2v+1) - \frac{1}{6}h^2. \tag{3.1}$$

Eqn (2.8) may be used again to eliminate the quadratic term in h; note that, in the original theory of Moffatt [3], this step is unnecessary since the quadratic h term in Q is absent. When the resulting equation is substituted into (3.1), further manipulation yields the following cubic equation for the free-surface velocity v:

$$(v-1)(4\beta(2v+1)^2 - 3(v+1) - 36\beta Q) = 6Q(6\beta(1-\beta Q) - 1). \tag{3.2}$$

In (3.2) we have, for the sake of brevity, made the substitution

$$\beta = \frac{1}{2}\gamma\cos\vartheta. \tag{3.3}$$

In the lubrication approximation, we require that $0 < h \ll 1$: this, we have seen, further implies that $\gamma \gg 1$. Under these conditions, we are therefore interested in solutions of (3.2) for which v is approximately unity. We now define the quadrivariate function F by

$$\begin{aligned} F(v,\vartheta,\gamma,Q) \equiv\ & (v-1)(2\gamma\cos\vartheta(2v+1)^2 - 3(v+1) - 18\gamma Q\cos\vartheta) \\ & +6Q\left(1 + 3\gamma\cos\vartheta\left(\frac{1}{2}\gamma Q\cos\vartheta - 1\right)\right). \end{aligned} \tag{3.4}$$

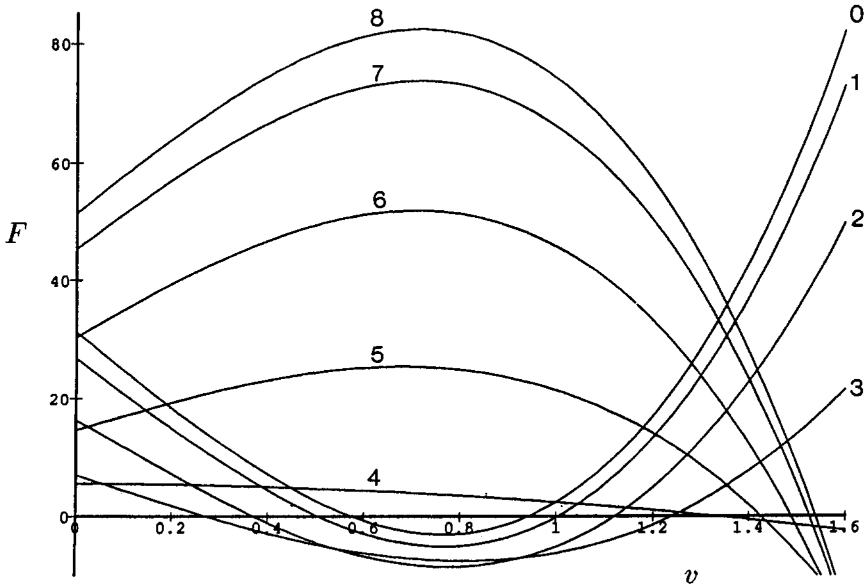

Fig. 1. The curves $F\left(v, \frac{n\pi}{8}, 5, 0.4\right), n = 0, \dots, 8$.

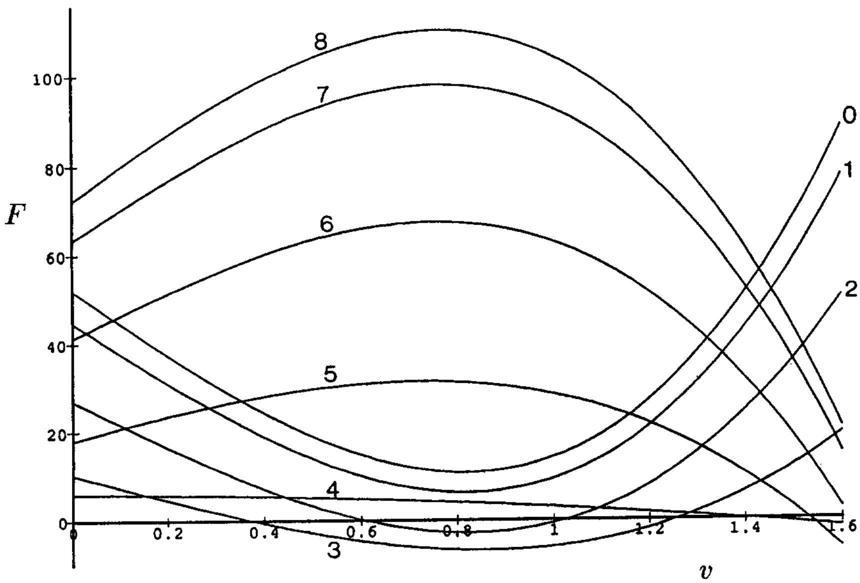

Fig. 2. The curves $F\left(v, \frac{n\pi}{8}, 5, 0.5\right), n = 0, \dots, 8$.

A steady solution to the problem exists if $F(v, \vartheta, \gamma, Q)$, defined by (3.4), has a root for: a v in the neighbourhood of 1; all $\vartheta \in [0, \pi[$ (the solution in $[\pi, 2\pi[$ is obtained by symmetry); all $\gamma > 0$; all $Q \in]0, 1[$.

It is informative now to examine F graphically, for fixed values of γ and Q. Figure 1 shows $F(v, \frac{n\pi}{8}, 5, 0.4)$, with separate curves indicated for $n = 0, \dots, 8$. By inspection, the cubic curve for $n = 0$ has the most-positive minimum, which minimum decreases until $n = 2$, whereafter it increases. At $n = 4$, corresponding to $\vartheta = \frac{\pi}{2}$, the curve is merely the inverted parabola

$$(1 - v)(1 + v) + 2Q = 0 \tag{3.5}$$

with (positive) root at $v = (1 + 2Q)^{1/2}$. Thereafter, the cubic curves possess maxima in the neighbourhood under inspection, which maxima increase until $n = 8$. Fig. 1 illustrates that F always has a root, in the neighbourhood of $v = 1$, for $\gamma = 5$ and $Q = 0.4$. Contrast this with Fig. 2, in which Q has been raised slightly to 0.5. Now the curves for $n = 0$ and $n = 1$ never assume a zero value in the required v-interval: $F(v, \vartheta, \gamma, Q)$ has no root and, therefore, no steady solution to the problem exists with this $Q - \gamma$ parameter pair. Physically, this may be interpreted as the fluid load exceeding that supportable under the angular velocity associated with $\gamma = 5$ (all other physical parameters being kept fixed).

By equating $F_\vartheta(v, \vartheta, \gamma, Q)$ to zero, we find that, as we traverse the free-surface from $\vartheta = 0$ to $\vartheta = \pi$ (with all other variables fixed), F assumes three stationary values at

$$\vartheta_1 = 0, \;\; \vartheta_2 = \cos^{-1}\left\{\frac{9Qv + (1 - v)(2v + 1)^2}{9Q^2\gamma}\right\} \quad \text{and} \quad \vartheta_3 = \pi. \tag{3.6}$$

Further analysis, the details of which are omitted for brevity, reveals that ϑ_1 corresponds to a saddle, ϑ_2 to a minimum, and ϑ_3 to a maximum. Generally speaking, $F(v, \vartheta_1, \gamma, Q)$ possesses a maximum-minimum; $F(v, \vartheta_2, \gamma, Q)$ a minimum-minimum, and $F(v, \vartheta_3, \gamma, Q)$ a maximum-maximum of $F(v, \vartheta, \gamma, Q)$ in the v-interval under scrutiny.

On equating $F_v(v, \vartheta, \gamma, Q)$ to zero, we find that, for a given pair of parameters γ and Q, F has stationary values of $F(v_{\pm k}, \vartheta_k, \gamma, Q), k = 1, 2, 3$, where

$$v_{\pm k}(\beta_k) = \frac{1}{16\beta_k}\{1 \pm (1 + 64\beta_k^2(1 + 3Q))^{1/2}\}, \tag{3.7}$$

β_k being obtained by substitution of ϑ_k into (3.3). It is essential to observe here that, if $\beta_k < 0$ (or > 0), v_{-k} (or v_{+k}) is the value of v required. Substitution of (3.6) and (3.7) into (3.4) now yields

$$\begin{aligned} F(v_{\pm k}, \vartheta_k, \gamma, Q) &= 36\beta_k^2 Q^2 - 4\beta_k + \frac{3}{4}(5Q + 3) \\ &\quad - \frac{1}{128\beta_k^2}\{1 \pm (1 + 64\beta_k^2(1 + 3Q))^{3/2}\}. \end{aligned} \tag{3.8}$$

We are now in possession of sufficient information to state simultaneous conditions for F always to possess a root in the relevant neighbourhood. We require that, for all γ and Q,

$$F(v_{+k}, \vartheta_1, \gamma, Q) \le 0 \quad \text{and} \quad F(v_{-k}, \vartheta_3, \gamma, Q) \ge 0. \tag{3.9}$$

Combining (3.8) with the first of (3.9), we obtain the inequality

$$9\gamma^2 Q^2 + \frac{3}{4}(5Q + 3) - \frac{1}{32\gamma^2} \le 2\gamma + \frac{1}{32\gamma^2}(1 + 16\gamma^2(1 + 3Q))^{3/2}. \tag{3.10}$$

The second condition in (3.9) yields a similar inequality, differing from (3.10) only in the following details: the direction of the inequality in (3.10) is reversed, and both terms on the right-hand side of (3.10) are negative.

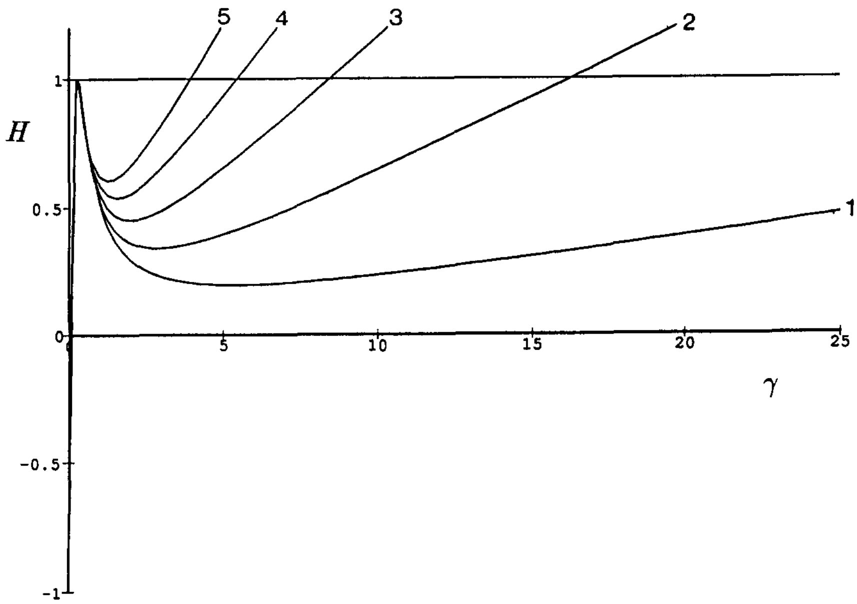

Fig. 3. The curves $H\left(\frac{n}{10},\gamma\right)$, $n = 1,\dots,5$.

If we now define the function $H(Q,\gamma)$ by

$$H(Q,\gamma) \equiv \frac{288\gamma^4 Q^2 + 24\gamma^2(5Q+3) - 1}{64\gamma^3 + (1+16\gamma^2(1+3Q))^{3/2}} \tag{3.11}$$

then (3.8), (3.10) and (3.11) combine to give the existence condition for steady flow as

$$|H(Q,\gamma)| \leq 1, \tag{3.12}$$

where, of course, Q and γ must lie in the ranges discussed above.

The algebraic complexity of condition (3.12) provokes a graphical study before further analysis is undertaken. Fig. 3 shows the curves $H(\frac{n}{10},\gamma)$, for $n = 1,\dots,5$. From (3.11), $H(Q,0) = -1$ and $H_\gamma(Q,0) \geq 0$ for all Q in the neighbourhood of $\gamma = 0$, so that condition (3.12) is not violated there.

A striking observation to be drawn from Fig. 3 is that there is a local maximum, at $\gamma = \gamma^*$, where $H(\gamma^*, Q)$ assumes a (computationally-verifiable) local-maximum of unity for *any* value of Q; the variation of abscissa of this maximum is only weakly dependent on Q, as can be seen in the expanded scale of Fig. 4. However, since this critical Stokes number, $\gamma^*(Q)$, lies outside the range of γ permitted by thin-film theory, it need cause us no concern: we are not bound to interpret the apparent metastability. On the other hand, it is encouraging that (3.12) is not violated until expected. We note in passing that, since $H_\gamma(Q,\gamma^*) = 0$ yields the complex implicit relationship

$$\frac{24\gamma^{*2}Q^2 + 5Q + 3}{4\gamma^* + (1+3Q)(1+16\gamma^{*2}(1+3Q))^{1/2}} = 1, \tag{3.13}$$

we cannot verify theoretically that $H(Q,\gamma^*) = 1$.

It is clear from Fig. 3 that, as Q is increased, so then must γ decrease in order to support the increased load. For a given physical system, inspection of (2.2) reveals that γ can be

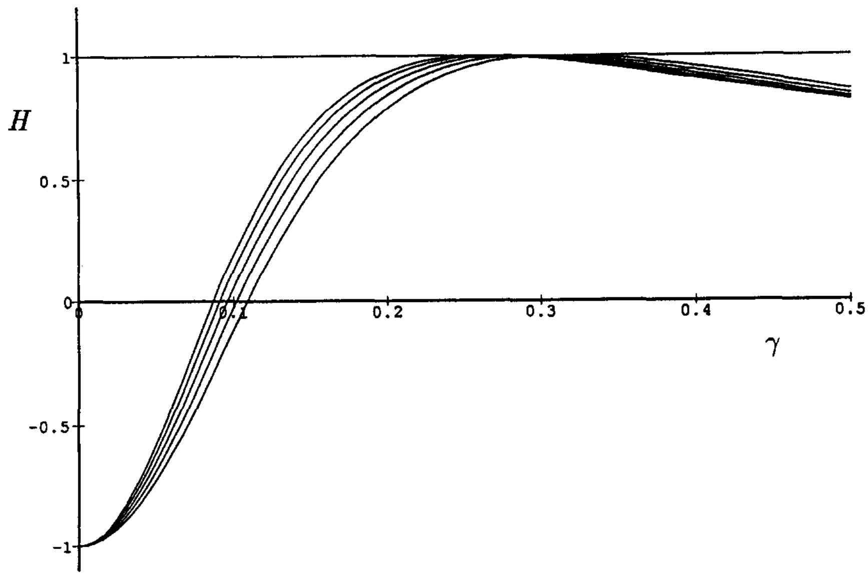

Fig. 4. Detail of Figure 3 near critical values $\gamma = \gamma^*$.

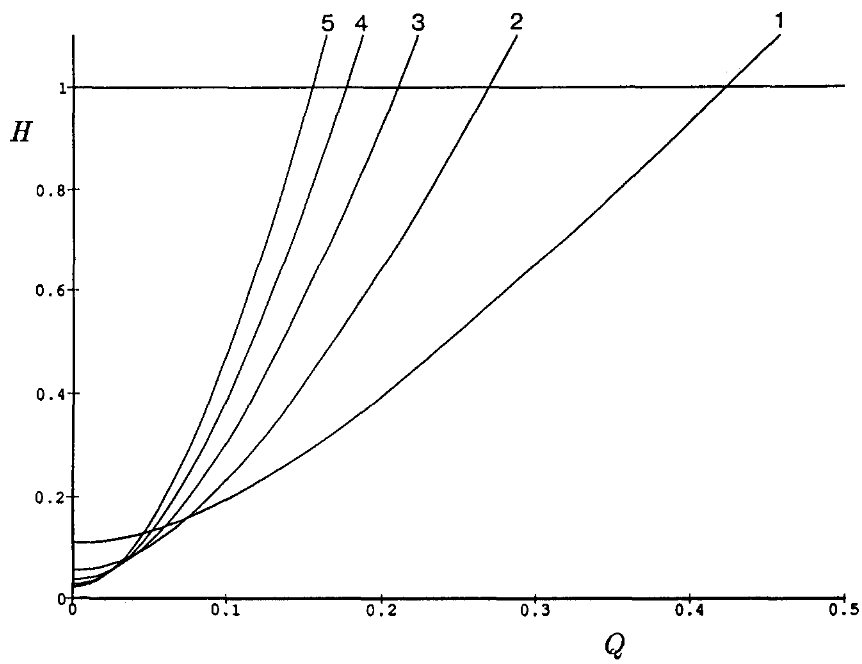

Fig. 5. The curves $H(Q, 5n), n = 1, \ldots, 5$.

decreased only by increasing the rotation rate ω. Thus (3.12) confirms the physically-intuitive result that a steady flow cannot exist as the fluid mass is increased, unless the cylinder rotates sufficiently rapidly to sustain that mass.

Conversely, Fig. 5 depicts the curves $H(Q, 5n)$, where $n = 1, \ldots, 5$. This clearly shows that, as γ is increased, so must Q decrease. In other words, (3.12) confirms that, as the cylinder

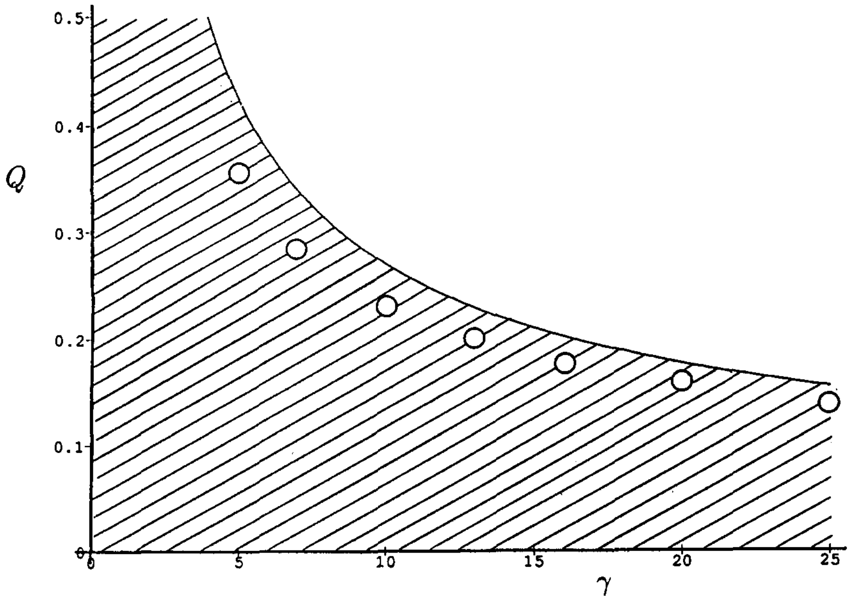

Fig. 6. The region in the $Q - \gamma$ plane (shaded) satisfying (3.12). Results of Hansen and Kelmanson [5] are indicated by discs.

rotation rate decreases (all other variables being fixed), the mass of fluid supportable must decrease accordingly; this again is in agreement with physical intuition.

Fig. 6 displays that section of the $Q - \gamma$ plane (shaded) for which (3.12) is satisfied. The small circles represent the results obtained, by Hansen and Kelmanson [5], from their numerical solution of the full, Stokes approximation; these are taken as the datum by which the results of thin-film approximations are compared. Since the results of Hansen and Kelmanson [5] lie appreciably within the locus given by (3.12), one concludes that thin-film theory consistently over-estimates the size of the existence region for the problem. As we shall see in the next section, a combination of (3.12) and the numerical results of Hansen and Kelmanson [5] permits us to suggest an approximate existence criterion for the *full Stokes approximation.*

4. Maximum fluid load

In the previous section, Figs. 3 and 4 confirmed that it is the upper inequality of (3.12),

$$H(Q,\gamma) \leq 1, \tag{4.1}$$

which is of interest. We now multiply (4.1) by the denominator in (3.11) and represent the resulting inequality in terms of a small-parameter expansion, centred on $Q = 0$; the details are omitted. If we neglect terms of order $O(Q^3)$ and higher, we obtain the inequality

$$9\gamma^2\left(1 - \frac{3}{\eta_1}\right)Q^2 + \frac{3}{4}(5 - 3\eta_1)Q - \left(2\gamma + \frac{1}{32\gamma^2}(1 + \eta_1^3)\right) \leq 0, \tag{4.2}$$

where

$$\eta_1 = (1 + 16\gamma^2)^{1/2}. \tag{4.3}$$

Table 1. A comparison of maximum fluid fluxes at different Stokes numbers.

γ	$\hat{Q}(\gamma)$	$\hat{Q}_s(\gamma)$	$\hat{Q}_{IE}(\gamma)$	$H(\hat{Q}_{IE},\gamma)$
3	0.610	0.677	0.485	0.792
5	0.424	0.449	0.355	0.803
7	0.339	0.352	0.285	0.792
10	0.270	0.277	0.230	0.793
13	0.229	0.234	0.200	0.811
16	0.202	0.206	0.175	0.796
20	0.177	0.180	0.160	0.848
25	0.155	0.157	0.135	0.790

We remark that, in the thin-film limit, (2.7) suggests that, for small h, Q will be accordingly small, justifying the neglect of order $O(Q^3)$ terms in (4.2). A tedious manipulation reveals that, if now

$$\eta_2 = 64\gamma^3 + 24\gamma^2 + 15 \tag{4.4}$$

and

$$\eta_3 = 256\gamma^4 - 192\gamma^3 - 208\gamma^2 - 17, \tag{4.5}$$

then $\hat{Q}_s = \hat{Q}_s(\gamma)$, the maximum value of Q which satisfies (4.2), is given explicitly by

$$\hat{Q}_s(\gamma) = \frac{\eta_1(3\eta_1 - 5) + (2\eta_1(\eta_1\eta_2 + \eta_3))^{1/2}}{24\gamma^2(\eta_1 - 3)}. \tag{4.6}$$

In (4.6), the subscript s distinguishes this series-derived result from the exact value, $\hat{Q}$, which satisfies (4.1). In Table 1 we present values of $\hat{Q}(\gamma)$ and $\hat{Q}_s(\gamma)$, the latter of which is easily calculated via (4.3) to (4.6). It is evident that, even over a large range of γ values, $\hat{Q}_s(\gamma)$ has a relative error of the order of 10^{-2}. Eqn (4.6) therefore provides an accurate, explicit formula for the greatest fluid flux which can exist on the cylinder at a given Stokes number.

The penultimate column in Table 1 are the values of $\hat{Q}_{IE}(\gamma)$ generated by the numerical, integral-equation technique of Hansen and Kelmanson [5]. The slight discrepancy between $\hat{Q}_{IE}(\gamma)$ and $\hat{Q}(\gamma)$ is, of course, due to the fact that the former are *more* likely to be accurate, since the integral-equation method solves the full, Stokes approximation of the equations of motion, whereas the present results are the solutions of only a thin-film approximation of the Stokes approximation. Here, the general agreement between $\hat{Q}_{IE}(\gamma)$ and $\hat{Q}(\gamma)$ encourages us that thin-film theory does indeed provide a reasonable approximation. The main advantage of the present analytical approach, over the work in [5], is that much theoretical, and, subsequently, computational, effort was required therein to obtain the values of $\hat{Q}_{IE}(\gamma)$: the (comparable) values of $\hat{Q}_s(\gamma)$ have been obtained at relatively little cost.

The final column in Table 1 shows the values of $H(\hat{Q}_{IE},\gamma)$; the results are remarkably consistent, given the total independence of the numerical results from the present theory. They would suggest that, for the full, Stokes approximation, an approximate existence criterion is

$$|H(Q,\gamma)| \leq 0.80. \tag{4.7}$$

Table 2. A comparison of maximum-supportable fluid loads at different Stokes numbers.

γ	$\hat{W}(\gamma)$	$\hat{W}_s(\gamma)$	$\hat{W}_{IE}(\gamma)$	$\lvert\varepsilon_{rel}\rvert$
3	3.377	3.788	3.415	1.09×10^{-1}
5	2.339	2.473	2.485	4.83×10^{-3}
7	1.891	1.962	1.943	9.78×10^{-3}
10	1.527	1.567	1.553	9.01×10^{-3}
13	1.312	1.339	1.354	1.11×10^{-2}
16	1.167	1.186	1.172	1.19×10^{-2}
20	1.031	1.045	1.092	4.30×10^{-2}
25	0.912	0.922	0.897	2.79×10^{-2}

Eqns (2.2) and (4.6) now enable us to derive an approximate expression for the maximum supportable load. We find that, if η_1, η_2 and η_3 are as given in (4.3) to (4.5), and if

$$\lambda_1 = \eta_1(3\eta_1 - 5), \tag{4.8}$$

$$\lambda_2 = 2\eta_1(\eta_1\eta_2 + \eta_3), \tag{4.9}$$

$$\lambda_3 = 24\gamma^2(\eta_1 - 3), \tag{4.10}$$

$$\tau_1 = \lambda_1(\lambda_1^2 - \lambda_1\lambda_3 + 2\lambda_3^2) + \lambda_2(3\lambda_1 - \lambda_3), \tag{4.11}$$

$$\tau_2 = 3\lambda_1^2 - 2\lambda_1\lambda_3 + 2\lambda_3^2 + \lambda_2, \tag{4.12}$$

then

$$\hat{W}_s(\gamma) = \frac{\tau_1 + \tau_2\lambda_2^{1/2}}{\lambda_3^3}\pi \tag{4.13}$$

is an explicit expression for the approximate maximum load which can be supported at the Stokes number γ.

In Table 2 we present a comparison of maximum fluid loads supportable at different Stokes numbers, γ. $\hat{W}(\gamma)$ is obtained by inserting $\hat{Q}(\gamma)$, given implicitly by $H(\hat{Q}, \gamma) = 1$, into (2.10); $\hat{W}_s(\gamma)$ is given by (4.13); $\hat{W}_{IE}(\gamma)$ is the maximum load found from the integral-equation technique of Hansen and Kelmanson [5]; the last column shows the absolute relative error $|1 - \hat{W}_s/\hat{W}_{IE}|$. As expected, agreement between $\hat{W}(\gamma)$ and $\hat{W}_s(\gamma)$ is excellent, but note that both agree, to a relative error of order $O(10^{-2})$, with the critical load, $\hat{W}_{IE}(\gamma)$, obtained via the integral-equation technique. Thus, Table 2 indicates that (4.13) predicts, to within a relative error of order $O(10^{-2})$, the maximum load supportable. The only case in which this bound is exceeded is that for $\gamma = 3$, in which case the thin-film condition $\gamma \gg 1$ is not satisfied.

Fig. 7 shows a plot of $\hat{W}_s(\gamma)$, with $\hat{W}_{IE}(\gamma)$ results superimposed for comparison: the agreement is excellent over a wide range of γ values, even those which correspond to flows outside the expected range of validity of thin-film theory. Also shown, for comparison, is the (dashed) curve $\hat{W}_s = 1.057 \times \frac{4\pi}{3}\gamma^{-1/2}$, which is the maximum load predicted by the theory of Moffatt [3]: the agreement between theories is clearly strong for $\gamma \gg 1$, but the present theory is evidently also applicable to flows for which γ is of order $O(1)$.

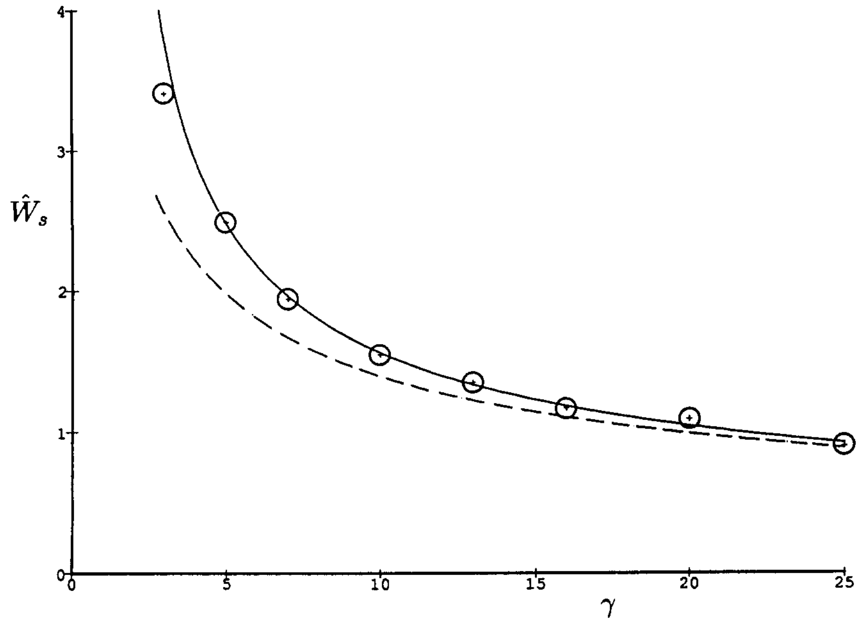

Fig. 7. The maximum-supportable fluid load, as given by (4.13). Results of Hansen and Kelmanson [5] are indicated by discs. Dashed line shows results of Moffatt [3].

5. Minimum angular velocity

In the experiments of Moffatt [3], γ values of order 10–20 were typical. In roller-coating industrial problems (see Thompson [6]), Stokes numbers of 30–50 are not uncommon. It is therefore of interest to derive large-γ formulae to determine the maximum value of γ, $\hat{\gamma}(Q)$, which can support a given load Q. This yields an explicit formula for the critical angular velocity of the cylinder, below which the fluid must drain off.

Since $\gamma \gg 1$, we substitute $\varepsilon = 1/\gamma$ into (4.1), expand in powers of ε, and neglect terms of order $O(\varepsilon^3)$ and higher; the details are tedious and therefore omitted. We obtain the inequality

$$\frac{3}{4}(5Q+3)\varepsilon^2 - 2(1+(1+3Q)^{3/2})\varepsilon + 9Q^2 \leq 0. \tag{5.1}$$

If now

$$\xi_1 = (1+3Q)^{1/2} \tag{5.2}$$

and

$$\xi_2 = 9Q^2 - 12Q - 8, \tag{5.3}$$

then, $\gamma = \hat{\gamma}_s(Q)$, the maximum (positive) value of $\gamma = 1/\varepsilon$ satisfying (5.1), is given explicitly by

$$\hat{\gamma}_s(Q) = \frac{3(5Q+3)}{2(2(1+\xi_1^3) - \xi_1(8\xi_1 - \xi_2)^{1/2})}. \tag{5.4}$$

In Table 3 are presented the results $\hat{\gamma}(Q)$, obtained from eqn (4.1), and $\hat{\gamma}_s(Q)$, obtained using the explicit result (5.4); the agreement between results is excellent. Relative errors are

Table 3. A comparison of minimum Stokes numbers which can sustain a given fluid load.

Q	$\check{\gamma}(Q)$	$\check{\gamma}_s(Q)$
0.1	54.628	54.627
0.2	16.290	16.288
0.3	8.447	8.442
0.4	5.450	5.442
0.5	3.947	3.937

2.5×10^{-3} at $Q = 0.5$, diminishing to 2×10^{-5} at $Q = 0.1$. An unexpected, yet welcome, feature of the results in Table 3 is that the approximation (5.4) seems to be remarkably accurate even for γ of order $O(1)$. As intuition suggests, high Stokes numbers can sustain only small fluid loads. The minimum value of ω, $\check{\omega}$, which can sustain (steadily) a prescribed load Q is therefore given by lubrication theory as

$$\check{\omega} = \frac{\rho g a}{\check{\gamma}_s(Q)\mu}. \tag{5.5}$$

For thin films, i.e. $0 < Q \ll 1$, (5.2) to (5.5) yield the explicit expression

$$\check{\omega} \cong \frac{9\rho g a}{4\mu} Q^2 \left[1 - \frac{9}{4}Q + \frac{297}{64}Q^2\right] \tag{5.6}$$

for the minimum angular velocity. Since, for a thin film, (2.7) implies that $0 < Q < 1$, a non-maximal version of (2.10) implies that $0 < W < 1$. Expanding Q in powers of W and equating coefficients therefore yields

$$Q = \frac{1}{2}\frac{W}{\pi} + \frac{1}{8}\left[\frac{W}{\pi}\right]^2 - \frac{5}{128}\left[\frac{W}{\pi}\right]^4 + O(W^5), \tag{5.7}$$

which, after substituting into (5.6), gives

$$\check{\omega} \cong \frac{9\rho g a}{16\mu}\left[\frac{W}{\pi}\right]^2 \left\{1 - \frac{5}{8}\left[\frac{W}{\pi}\right] + \frac{97}{256}\left[\frac{W}{\pi}\right]^2\right\}. \tag{5.8}$$

From an experimental point of view, (5.8) is a more useful formula than (5.6) since W can be measured more easily than Q. If V_M is the measured volume of fluid deposited on a cylinder of length L, and $V = V_M/\pi a^2 L$, then the expression

$$\check{\omega} \cong \frac{9ga}{16\nu} V^2 \left\{1 - \frac{5}{8}V + \frac{97}{256}V^2\right\} \tag{5.9}$$

can be used to investigate experimentally the minimum angular velocity.

6. An experiment

The author had access to a desk-top, variable-speed jeweller's lathe, whose rotation rate was easily calibrated. One end of a brass rod, of diameter 20 mm and length 30 cm, was centred in the four-jaw chuck; the other end was countersunk centrally and supported by a centreing-cone. A 20 cm section of the rod (clear of both chuck and support) was turned down to a diameter of 12 mm, care being taken to ensure that the 4 mm step, at each end of the narrower section, was at right angles to the axis of the cylinder. Thus, in formula (5.9), $a = 6 \times 10^{-3}$ m and $L = 0.2$ m (the latter, of course, being contained implicitly). The surface of the cut region was abraded with progressively finer grades of emery paper, finishing with a flat water-of-Ayr stone. With the rod still in situ (so as not to disturb the exact, central position for the fluid experiment), the rod was cleaned with methylated spirits and dried.

Three engine oils were used to test the theory of the previous section: an SAE 30 monograde oil; an SAE 50 monograde oil; "Gold X", an SAE 15W/50 multigrade oil. Since the author did not have access to viscometric equipment, and since accurate (industrially-provided) viscosities are available only at temperatures of 40°C and 100°C, the formulae of Andrade [7] were used to evaluate kinematic viscosities at the experimental temperature of 18°C. In an obvious notation, oil densities (in kg m^{-3} at 18°C) were $\rho_{30} = 890$, $\rho_{50} = 896$ and $\rho_{GX} = 880$; oil viscosities (in $10^{-4}\,m^2\,s^{-1}$ were calculated to be $\nu_{30} = 2.26$, $\nu_{50} = 6.35$ and $\nu_{GX} = 3.78$. At similar temperatures, the viscosity of Shell Tellus R5 sewing-machine oil (used in Malone's [8] experiments) is $0.084 \times 10^{-4}\,m^2\,s^{-1}$ and that of the water-sugar mixture (used in Moffatt's [3] experiments) is approximately $70 \times 10^{-4}\,m^2\,s^{-1}$.

Using a calibrated pipette, measured volumes (in increments of 1 cm^3) of oil were deposited on the already-rotating cylinder, which was then left until the oil had spread along the 20 cm channel. (The cylinder length, coupled with the moderately-low rotation rates, was intended to preclude both end-effects and the onset of ribbing, i.e. periodic variations, in film thickness, along the cylinder. At higher rotation rates, Moffatt's [3] experiments demonstrate admirably the transition from steady flow, through ribbing, to complete breakdown of two-dimensional flow.) The cylinder rotation rate was then gradually reduced until the first (few) drip(s) of fluid were witnessed to fall from the cylinder, at which point the rotation rate was recorded. The experiment was repeated using different volumes of each oil on the cylinder.

The qualitative behaviour of results was the same for all three oils. The rotation rate of the cylinder was gradually decreased until, as expected, a line of lobes began to form on the underside of the cylinder, directly below, and parallel with, the axis of the cylinder. The presence of a line of connected lobes, as opposed to a single, linear lobe along the cylinder, bears witness to the fact that our (moderately-crude) experiment could not produce a genuinely two-dimensional flow, particularly when fluid shedding was imminent. Moreover, at the higher rotation rates (corresponding to larger film thicknesses), ribbing did occur to some extent.

Given the crudity of the experiment, together with the fact that viscosities were calculated, the comparison with theory was surprisingly good. This may perhaps support Andrade's [7] claim that his formulae remained accurate over large variations in temperature (and, therefore, viscosity). However, crude as these experiments are, the author has been unable to trace any previous experimental work which addresses the particular aspects of the present investigation, and so the results reported hereafter are new.

In Fig. 8, the three continuous curves are given by expression (5.9), with ν_{30}, ν_{50} and ν_{GX} inserted; the symbols are the results of our experiments. It is evident that the theory consistently underestimates the values of $\check{\omega}$, for all three oils, i.e. load shedding always occurs at a rotation rate higher than predicted. This is possibly due to factors influenced by surface tension: at the lower rotation rates, the tension-driven accumulation of the aforementioned lobes clearly violates our assumption of two-dimensional flow; at the higher rotation rates, the same violation occurs due to ribbing. Results at low rotation rates were not achievable due to

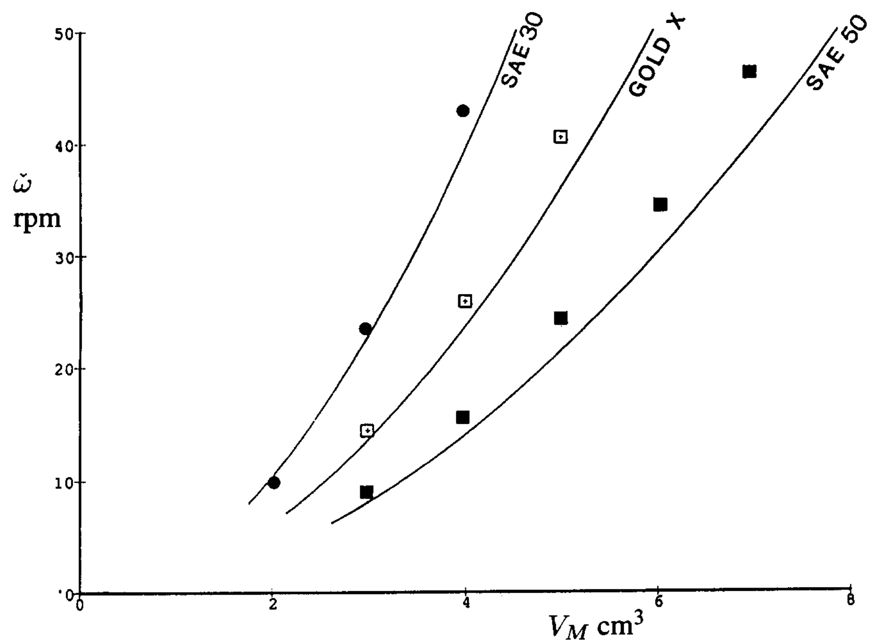

Fig. 8. The minimum angular velocity, as given by (5.9). Results of experiments are indicated by: ● SAE 30 oil; ■ SAE 50 oil; □ "Gold X" oil. Symbols indicate the mean value over five series of measurements.

the fact that small volumes of the oil tended not to spread evenly along the cylinder. We note in passing that, for all the results presented, the maximum film thickness was of the order of 1 mm (for the SAE 50 oil), and the maximum Reynolds number was approximately unity (for the SAE 30 oil). In all results, γ was greater than approximately 14.

We return briefly to the observation of an accumulation of lobes along the underside of the cylinder. The volumes of oil on the cylinder imply that the film-thicknesses-to-length ratio was, in every experiment performed, of the order of 10^{-3}: the breakdown in the flow's two-dimensionality is therefore unlikely to be due to end-effects. More likely, it is because the mechanism immediately preceding fluid shedding causes fluid to accumulate, through surface tension effects and gravity, into preferred regions on the underside of the cylinder. Furthermore, at the onset of fluid shedding, it is conceivable that the film thickness on the upper edge of the cylinder has become so reduced that, at particularly-low rotation rates, the flow there becomes starved and a dynamic wetting line is in evidence. Such effects have so far been neglected by our theory, and their consideration is beyond the scope of the present paper.

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